1. Using the **Principle of Mathematical Induction**, prove that $\sum_{r=1}^{n}(2r-1) = n^2$ for all $n \in \mathbb{Z}^+$.

For $n = 1$, L.H.S. = $2 \times 1 - 1 = 1$ and R.H.S. = $1^2 = 1$ 5

$\therefore$ The result is true for $n = 1$.

Take any $p \in \mathbb{Z}^+$ and assume that the result is true for $n = p$.

i.e. $\sum_{r=1}^{p} (2r-1) = p^2$. 5

Now $\sum_{r=1}^{p+1} (2r-1) = \sum_{r=1}^{p} (2r-1) + \big(2(p+1)-1\big)$ 5

$= p^2 + (2p + 1)$

$= (p+1)^2$. 5

Hence, if the result is true for $n = p$, then it is true for $n = p + 1$. We have already proved that the result is true for $n = 1$.

Hence, by the Principle of Mathematical Induction, the result is true for all $n \in \mathbb{Z}^+$. 5

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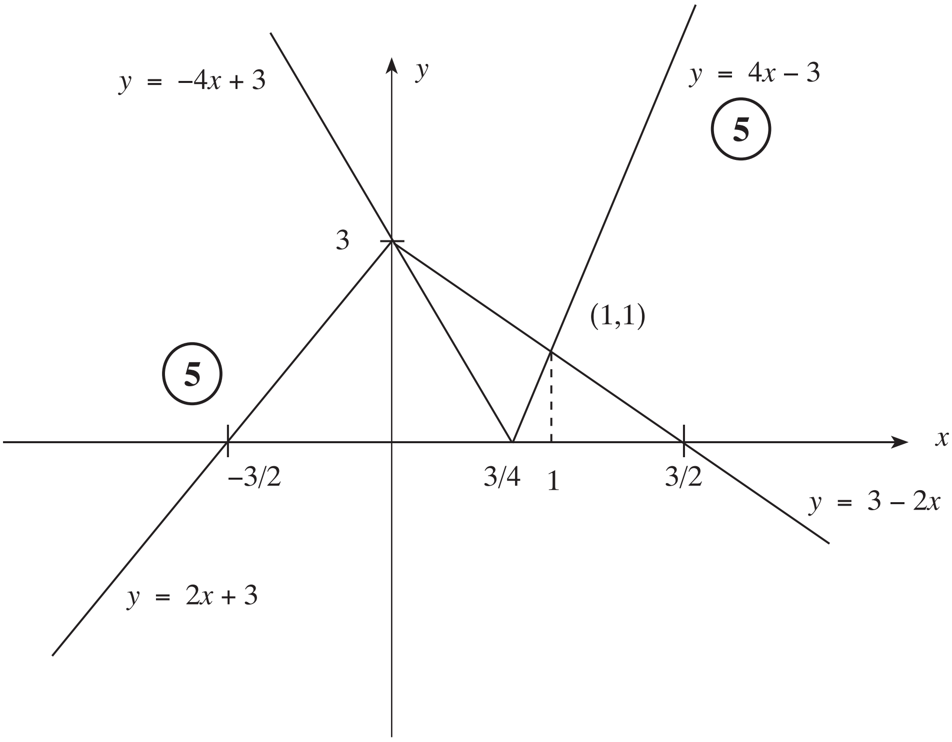
2. Sketch the graphs of $y=|4x-3|$ and $y=3-2|x|$ in the same diagram.
Hence or otherwise, find all real values of x satisfying the inequality $|2x-3|+|x|<3$.



At the point of intersections of the graphs

$$4x - 3 = 3 - 2x \Rightarrow x = 1 \quad 5$$

$$-4x + 3 = 3 + 2x \Rightarrow x = 0$$

From the graphs, we have,

$|4x - 3| < 3 - 2|x| \quad \Leftrightarrow \quad 0 < x < 1$

$\therefore \quad |4x - 3| + |2x| < 3 \quad \Leftrightarrow \quad 0 < x < 1$

Replacing x by $\frac{x}{2}$, we get

$$|2x - 3| + |x| < 3 \quad \Leftrightarrow \quad 0 < x < 2. \quad 5$$

Hence, the set of all values of x satisfying

$|2x - 3| + |x| < 3$ is $\{x : 0 < x < 2\}$. 5

25

Aliter

For the graphs 5 + 5 , as before.

Aliter for values of x

$|2x-3| + |x| < 3$

Case (i) $x \leq 0$:

Then $|2x-3| + |x| < 3 \Leftrightarrow -2x+3-x < 3$

$\Leftrightarrow 3x > 0$

$\Leftrightarrow x > 0$

Hence, in this case, no solutions exist.

Case (ii) $0 < x \leq \frac{3}{2}$

Then $|2x-3| + |x| < 3 \Leftrightarrow -2x+3+x < 3$

$\Leftrightarrow x > 0$

Hence, in this case, the solutions are the values of x satisfying $0 < x \leq \frac{3}{2}$.

Case (iii) $x > \frac{3}{2}$

Then $|2x-3| + |x| < 3 \Leftrightarrow 2x-3+x < 3$

$\Leftrightarrow 3x < 6$

$\Leftrightarrow x < 2$

Hence, in this case, the solutions are the values of x satisfying $\frac{3}{2} < x < 2$.

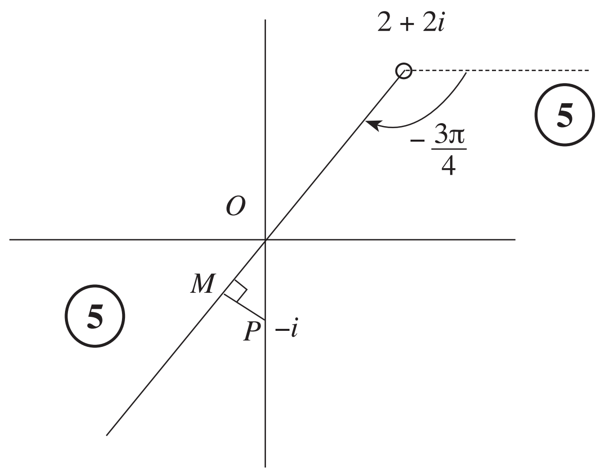
All 3 cases with correct solutions	10
Any 2 cases with correct solutions	5

Hence, over all, the solutions are values of x satisfying $0 < x < 2$. 5

$\boxed{25}$

3. Sketch, in an Argand diagram, the locus of the points that represent complex numbers z satisfying $\text{Arg}(z-2-2i) = -\frac{3\pi}{4}$.

Hence or otherwise, find the minimum value of $|i\bar{z}+1|$ such that $\text{Arg}(z-2-2i) = -\frac{3\pi}{4}$.



Note that

$|i\bar{z}+1| = |i(\bar{z}-i)| = |\bar{z}-i| = |\overline{z+i}|$

$= |z+i|$

$= |z-(-i)|$ 5

Hence, the minimum of $|i\bar{z}+1|$ is equal to PM. 5

Now, PM $= 1 . \sin\frac{\pi}{4} = \frac{1}{\sqrt{2}}$. 5

25

4. Show that the coefficient of x^6 in the binomial expansion of $\left(x^3 + \frac{1}{x^2}\right)^7$ is 35.

Show also that there does not exist a term independent of x in the above binomial expansion.

$$\left(x^3 + \frac{1}{x^2}\right)^7 = \sum_{r=0}^{7} {}^7C_r \, (x^3)^r \left(\frac{1}{x^2}\right)^{7-r} \quad 5$$

$$= \sum_{r=0}^{7} {}^7C_r \, x^{5r-14}$$

x^6 : $5r - 14 = 6 \Leftrightarrow r = 4.$ 5

$\therefore$ The coefficient of $x^6 = {}^7C_4 = 35$ 5

For the above expansion to have a term independent of x, we must have

$5r - 14 = 0.$ 5

This is not possible as $r \in \mathbb{Z}^+$. 5

25

5. Show that $\lim\limits_{x \to 3} \dfrac{\sqrt{x-2}-1}{\sin(\pi(x-3))} = \dfrac{1}{2\pi}$.

$$\lim_{x \to 3} \frac{\sqrt{x-2} - 1}{\sin(\pi(x-3))} = \lim_{x \to 3} \frac{\sqrt{x-2} - 1}{\sin(\pi(x-3))} \cdot \frac{(\sqrt{x-2} + 1)}{(\sqrt{x-2} + 1)} \quad 5$$

$$= \lim_{x-3 \to 0} \frac{x-3}{\sin(\pi(x-3))} \cdot \lim_{x \to 3} \frac{1}{(\sqrt{x-2} + 1)} \quad 5$$

$$= \lim_{x-3 \to 0} \frac{1}{\dfrac{\sin(\pi(x-3))}{\pi(x-3)}} \cdot \frac{1}{\pi} \cdot \frac{1}{2} \quad 5$$

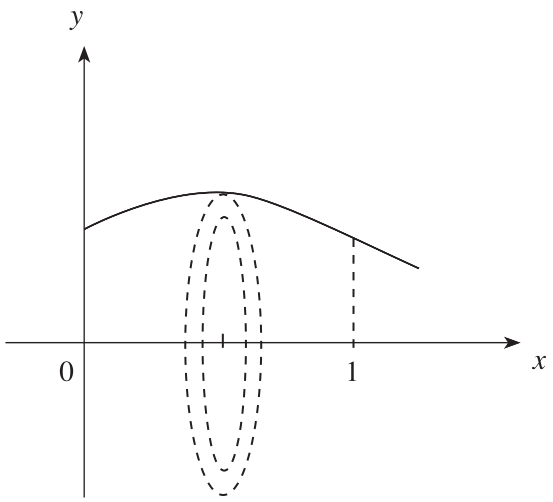
$$= 1 \cdot \frac{1}{\pi} \cdot \frac{1}{2}$$

5

$$= \frac{1}{2\pi} \quad 5$$

25

6. The region enclosed by the curves $y = \sqrt{\dfrac{x+1}{x^2+1}}$, $x=0$, $x=1$ and $y=0$ is rotated about the x-axis through 2π radians. Show that the volume of the solid thus generated is $\dfrac{\pi}{4}(\pi + \ln 4)$.



The volume generated $= \displaystyle\int_0^1 \pi \left(\sqrt{\frac{x+1}{x^2+1}}\right)^2 dx$ 5

$= \pi \left(\displaystyle\int_0^1 \frac{x}{x^2+1}\, dx + \int_0^1 \frac{1}{x^2+1}\, dx \right)$ 5

$= \pi \left(\dfrac{1}{2} \ln (x^2+1) \Big|_0^1 + \tan^{-1} x \Big|_0^1 \right)$ 5 + 5

$= \pi \left(\dfrac{1}{2} \ln 2 + \dfrac{\pi}{4} \right)$

$= \dfrac{\pi}{4} \left(\ln 4 + \pi \right)$ 5

25

7. Let C be the parabola parametrically given by $x = at^2$ and $y = 2at$ for $t \in \mathbb{R}$, where $a \neq 0$. Show that the equation of the normal line to the parabola C at the point $(at^2, 2at)$ is given by $y + tx = 2at + at^3$.

The normal line at the point $P \equiv (4a, 4a)$ on the parabola C meets this parabola again at a point $Q \equiv (aT^2, 2aT)$. Show that $T = -3$.

$x = at^2 \ , \ y = 2at$

$\frac{dx}{dt} = 2at \ , \ \frac{dy}{dt} = 2a$

$\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx} = 2a \cdot \frac{1}{2at} = \frac{1}{t}$ for $t \neq 0$. 5

$\therefore$ The slope of the normal line $= -t$

The equation of the normal at $(at^2, 2at)$ is

$y - 2at = -t\ (x - at^2)$

$y + tx = 2at + at^3$ 5 (This is valid for $t = 0$ also.)

$P \equiv (4a, 4a)$ on $C \Rightarrow t = 2$.

The normal line at P : $y + 2x = 4a + 8a = 12a$ 5

Since it meets C at $(aT^2, 2aT)$, we have

$2aT + 2aT^2 = 12a.$ 5

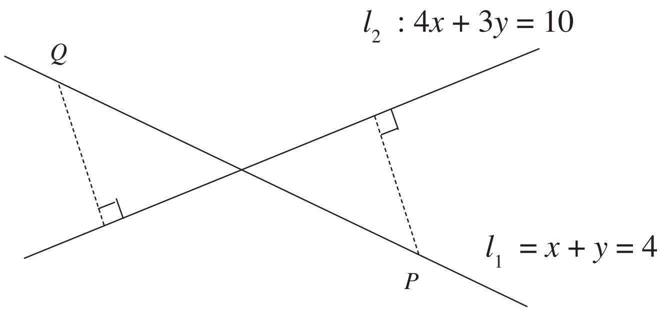
$\Leftrightarrow T^2 + T - 6 = 0 \Leftrightarrow (T - 2)(T + 3) = 0$

$\Leftrightarrow T = 2$ or $T = -3$

$\therefore T = -3$ 5

$\boxed{25}$

8. Let l_1 and l_2 be the straight lines given by $x + y = 4$ and $4x + 3y = 10$, respectively. Two distinct points P and Q are on the line l_1 such that the perpendicular distance from each of these points to the line l_2 is 1 unit. Find the coordinates of P and Q.



Any point on the line l_1 can be written in the form

$(t, 4 - t),\ t \in \mathbb{R}$. 5

Let $P \equiv (t_1, 4 - t_1)$

Perpendicular distance from P to l_2 $= \dfrac{|4t_1 + 3(4 - t_1) - 10|}{\sqrt{4^2 + 3^2}} = 1$ 5

$\therefore\ |t_1 + 2| = 5$

$\therefore\ t_1 = -7$ or $t_1 = 3$ 5

The coordinates of P and Q are

$(-7, 11)$ and $(3, 1)$. 5 + 5

25

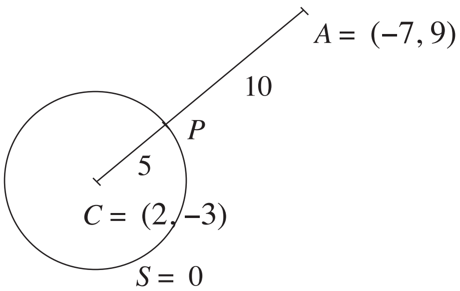
9. Show that the point $A \equiv (-7, 9)$ lies outside the circle $S \equiv x^2 + y^2 - 4x + 6y - 12 = 0$.
Find the coordinates of the point on the circle $S = 0$ nearest to the point A.

The centre C of $S = 0$ is $(2, -3)$. 5

The radius R of $S = 0$ is $\sqrt{4 + 9 + 12} = \sqrt{25} = 5.$ 5

$CA^2 = 9^2 + 12^2 = 15^2 \Rightarrow CA = 15 > R = 5.$ 5

$\therefore$ Point A lies outside the given circle.



The point on the circle $S = 0$ nearest to point A is the point P at which CA meets $S = 0$.

Note that $CP : PA = 5 : 10$

$= 1 : 2$ 5

$$\therefore \; P \equiv \left(\frac{2 \times 2 + 1\,(-7)}{3} \, , \; \frac{2\,(-3) + 1 \times 9}{3} \right)$$

i.e. $P \equiv (-1, \; 1)$ 5

25

10. Let $t = \tan\frac{\theta}{2}$ for $\theta \neq (2n+1)\pi$, where $n \in \mathbb{Z}$. Show that $\cos\theta = \frac{1-t^2}{1+t^2}$.

Deduce that $\tan\frac{\pi}{12} = 2-\sqrt{3}$.

$$\cos\theta = \cos^2\frac{\theta}{2} - \sin^2\frac{\theta}{2} \quad 5$$

$$= \frac{\cos^2\frac{\theta}{2} - \sin^2\frac{\theta}{2}}{\cos^2\frac{\theta}{2} + \sin^2\frac{\theta}{2}} = \frac{1 - \tan^2\frac{\theta}{2}}{1 + \tan^2\frac{\theta}{2}} \quad 5 \qquad \text{for } \theta \neq (2n+1)\pi.$$

$$= \frac{1-t^2}{1+t^2}$$

Let $\theta = \frac{\pi}{6}$. Then $\frac{\sqrt{3}}{2} = \frac{1-t^2}{1+t^2}$

5

$$\Rightarrow \sqrt{3}\,(1+t^2) = 2\,(1-t^2)$$

$$(2+\sqrt{3})\,t^2 = 2-\sqrt{3}$$

$$\therefore \quad t^2 = \frac{(2-\sqrt{3})}{(2+\sqrt{3})} \quad 5$$

$$= (2-\sqrt{3})^2$$

$$\Rightarrow t = \tan\frac{\pi}{12} = 2-\sqrt{3} \quad 5 \qquad \left(\because \tan\frac{\pi}{12} > 0\right)$$

25

11. (*a*) Let $p \in \mathbb{R}$ and $0 < p \leq 1$. Show that 1 is **not** a root of the equation $p^2x^2 + 2x + p = 0$.

Let α and β be the roots of this equation. Show that α and β are both real.
Write down $\alpha + \beta$ and $\alpha\beta$ in terms of p, and show that

$$\frac{1}{(\alpha - 1)} \cdot \frac{1}{(\beta - 1)} = \frac{p^2}{p^2 + p + 2}.$$

Show also that the quadratic equation whose roots are $\frac{\alpha}{\alpha - 1}$ and $\frac{\beta}{\beta - 1}$ is given by $(p^2 + p + 2)x^2 - 2(p + 1)x + p = 0$ and that both of these roots are positive.

(*b*) Let c and d be two **non-zero** real numbers and let $f(x) = x^3 + 2x^2 - dx + cd$. It is given that $(x - c)$ is a factor of $f(x)$ and that the remainder when $f(x)$ is divided by $(x - d)$ is cd. Find the values of c and d.

For these values of c and d, find the remainder when $f(x)$ is divided by $(x + 2)^2$.

(*a*) Suppose that 1 is a root of $p^2x^2 + 2x + p = 0$.

By substituting $x = 1$, we must have $p^2 + 2 + p = 0$. 5

This is impossible, as $p > 0$ implies that $p^2 + 2 + p > 0$. 5

$\therefore$ 1 is not a root of $p^2x^2 + 2x + p = 0$ $\boxed{10}$

The discriminant $\Delta = 2^2 - 4p^2 \, . \, p$ 10

$= 4(1 - p^3)$

$\geq 0 \; (\because \; 0 < p \leq 1)$ 5

$\therefore$ α and β are both real. 5 $\boxed{20}$

$\alpha + \beta = -\frac{2}{p^2}$ and $\alpha\beta = \frac{1}{p}$ 5 + 5

Now,

$$\frac{1}{(\alpha - 1)} \cdot \frac{1}{(\beta - 1)} = \frac{1}{(\alpha\beta - (\alpha + \beta) + 1)} \quad 5$$

$$= \frac{1}{\frac{1}{p} + \frac{2}{p^2} + 1}$$

$$= \frac{p^2}{p^2 + p + 2}. \quad 5$$

$\boxed{20}$

Now

$$\frac{\alpha}{\alpha-1}+\frac{\beta}{\beta-1}=\frac{\alpha(\beta-1)+\beta(\alpha-1)}{(\alpha-1)(\beta-1)}$$

$$=\frac{2\alpha\beta-(\alpha+\beta)}{(\alpha-1)(\beta-1)} \quad 5$$

$$=\left(\frac{2}{p}+\frac{2}{p^2}\right)\cdot\frac{p^2}{p^2+p+2} \quad 5$$

$$=\frac{2(p+1)}{p^2}\cdot\frac{p^2}{p^2+p+2}$$

$$=\frac{2(p+1)}{p^2+p+2} \quad 5$$

and

$$\frac{\alpha}{\alpha-1}\cdot\frac{\beta}{\beta-1}=\frac{\alpha\beta}{(\alpha-1)(\beta-1)}$$

$$=\frac{1}{p}\cdot\frac{p^2}{p^2+p+2}$$

$$=\frac{p}{p^2+p+2}\,. \quad 5$$

Hence, the required quadratic equation is given by

$$x^2-\frac{2(p+1)}{p^2+p+2}\,x+\frac{p}{p^2+p+2}=0 \quad 10$$

$$\Rightarrow \quad (p^2+p+2)\,x^2-2\,(p+1)\,x+p=0 \quad 5$$

$\boxed{35}$

Moreover, note that $\dfrac{\alpha}{(\alpha-1)}$ and $\dfrac{\beta}{(\beta-1)}$ are both real,

$$\frac{\alpha}{(\alpha-1)}+\frac{\beta}{(\beta-1)}=\frac{2(p+1)}{p^2+p+2}>0, \quad (\because\ p>0)\,,$$

5

and $$\frac{\alpha}{(\alpha-1)}\cdot\frac{\beta}{(\beta-1)}=\frac{p}{p^2+p+2}>0, \quad (\because\ p>0)\,.$$

Hence, both of these roots are possitive. 5

$\boxed{10}$

(**b**) $f(x) = x^3 + 2x^2 - dx + cd$

Since $(x-c)$ is a factor, $f(c) = 0$. 5

$\Rightarrow c^3 + 2c^2 - dc + cd = 0$ 5

$\Rightarrow c^2 (c+2) = 0$

$\Rightarrow c = -2 \quad (\because c \neq 0)$ 5

Since, when $f(x)$ is divided by $(x - d)$, the remainder is cd, we have

$$f(d) = cd.$$ 5

$\Rightarrow d^3 + 2d^2 - d^2 + cd = cd$ 5

$\Rightarrow d^3 + d^2 = 0$

$\Rightarrow d^2 (d+1) = 0$

$\Rightarrow d = -1 \quad (\because d \neq 0)$ 5

$\therefore \quad c = -2$ and $d = -1$. **30**

$f(x) = x^3 + 2x^2 + x + 2.$

Let $Ax + B$ be the remainder, when $f(x)$ is divided by $(x+2)^2$.

Then $f(x) \equiv (x+2)^2 Q(x) + (Ax + B)$, where $Q(x)$ is a polynomial of degree 1.

So, $x^3 + 2x^2 + x + 2 \equiv (x+2)^2 Q(x) + Ax + B$. 5

Substituting $x = -2$, we obtain $0 = -2A + B$. 5

By differentiating, we have

$$3x^2 + 4x + 1 = (x+2)^2 Q'(x) + 2Q(x)(x+2) + A.$$ 5

Again by substituting $x = -2$, we obtain

$$12 - 8 + 1 = A$$ 5

$\therefore \quad A = 5$ and $B = 10$

Hence the remainder is $5x + 10$. 5 **25**

Aliter

By long division we have,

$$
\begin{array}{r|l}
 & \quad x \ - \ 2 \\
\cline{2-2}
x^2 + 4x + 4 & x^3 + 2x^2 + x + 2 \\
 & \underline{x^3 + 4x^2 + 4x} \\
 & \quad -2x^2 - 3x + 2 \\
 & \quad \underline{-2x^2 - 8x - 8} \\
 & \qquad\qquad 5x + 10.
\end{array}
$$

15

$$x^3 + 2x^2 + x + 2 \equiv (x^2 + 4x + 4)(x - 2) + (5x + 10)$$

$\therefore$ Required remainder is $5x + 10$. 10

$\boxed{25}$

12. (*a*) Let P_1 and P_2 be the two sets given by $\{A,B,C,D,E,1,2,3,4\}$ and $\{F,G,H,I,J,5,6,7,8\}$ respectively. It is required to form a password consisting of 6 elements taken from $P_1 \cup P_2$ of which 3 are different letters and 3 are different digits. In each of the following cases, find the number of different such passwords that can be formed:

(i) all 6 elements are chosen only from P_1,

(ii) 3 elements are chosen from P_1 and the other 3 elements from P_2.

(*b*) Let $U_r = \dfrac{1}{r(r+1)(r+3)(r+4)}$ and $V_r = \dfrac{1}{r(r+1)(r+2)}$ for $r \in \mathbb{Z}^+$.

Show that $V_r - V_{r+2} = 6U_r$ for $r \in \mathbb{Z}^+$.

Hence, show that $\displaystyle\sum_{r=1}^{n} U_r = \frac{5}{144} - \frac{(2n+5)}{6(n+1)(n+2)(n+3)(n+4)}$ for $n \in \mathbb{Z}^+$.

Let $W_r = U_{2r-1} + U_{2r}$ for $r \in \mathbb{Z}^+$.

Deduce that $\displaystyle\sum_{r=1}^{n} W_r = \frac{5}{144} - \frac{(4n+5)}{24(n+1)(n+2)(2n+1)(2n+3)}$ for $n \in \mathbb{Z}^+$.

Hence, show that the infinite series $\displaystyle\sum_{r=1}^{\infty} W_r$ is convergent and find its sum.

(*a*) $P_1 = \{A, B, C, D, E, 1, 2, 3, 4\}$ and $P_2 = \{F, G, H, I, J, 5, 6, 7, 8\}$

(i) The number of different ways of choosing 3 different letters and 3 different digits from $P_1 = {}^5C_3 \cdot {}^4C_3$ 10

Hence the number of passwords that can be formed by choosing all 6 elements from P_1

$= {}^5C_3 \cdot {}^4C_3 \cdot 6!$ 5

$= 28800$ 5

20

(ii)

Different ways of selecting				Number of Passwords	
from P_1		from P_2			
Letters	Digits	Letters	Digits		
3	–	–	3	${}^5C_3 \cdot {}^4C_3 \cdot 6! = 28800$	10
2	1	1	2	${}^5C_2 \cdot {}^4C_1 \cdot {}^5C_1 \cdot {}^4C_2 \cdot 6! = 864000$	10
1	2	2	1	${}^5C_1 \cdot {}^4C_2 \cdot {}^5C_2 \cdot {}^4C_1 \cdot 6! = 864000$	10
–	3	3	–	${}^4C_3 \cdot {}^5C_3 \cdot 6! = 28800$	10

Hence, the number of different passwords that can be formed by choosing 3 elements

from P_1 and the other 3 elements from P_2 = 28800 + 864000 + 864000 + 28800 = 1785600

10

$\boxed{50}$

(b) $U_r = \dfrac{1}{r(r+1)(r+3)(r+4)}$ and $V_r = \dfrac{1}{r(r+1)(r+2)}$; $r \in \mathbb{Z}^+$.

Then,

$$V_r - V_{r+2} = \frac{1}{r(r+1)(r+2)} - \frac{1}{(r+2)(r+3)(r+4)}$$ 5

$$= \frac{(r+3)(r+4) - r(r+1)}{r(r+1)(r+2)(r+3)(r+4)}$$

$$= \frac{6(r+2)}{r(r+1)(r+2)(r+3)(r+4)}$$ 5

$$= 6U_r$$ 5

$\boxed{15}$

Now note that,

$r = 1; \quad 6U_1 = V_1 - \cancel{V_3},$

$r = 2; \quad 6U_2 = V_2 - \cancel{V_4},$ 10

$r = 3; \quad 6U_3 = \cancel{V_3} - V_5,$

$r = 4; \quad 6U_4 = \cancel{V_4} - V_6,$

$\vdots \quad \vdots \quad \vdots$

$r = n-3; \quad 6U_{n-3} = V_{n-3} - \cancel{V_{n-1}}$

$r = n-2; \quad 6U_{n-2} = V_{n-2} - \cancel{V_n}$ 10

$r = n-1; \quad 6U_{n-1} = \cancel{V_{n-1}} - V_{n+1}$

$r = n; \quad 6U_n = \cancel{V_n} - V_{n+2}$

$$\therefore\ 6\sum_{r=1}^{n} U_r = V_1 + V_2 - V_{n+1} - V_{n+2} \quad 10$$

$$= \frac{1}{6} + \frac{1}{24} - \frac{1}{(n+1)(n+2)(n+3)} - \frac{1}{(n+2)(n+3)(n+4)} \quad 5$$

$$= \frac{5}{24} - \frac{2n+5}{(n+1)(n+2)(n+3)(n+4)}$$

$$\therefore \sum_{r=1}^{n} U_r = \frac{5}{144} - \frac{2n+5}{6(n+1)(n+2)(n+3)(n+4)} \quad 5$$

$\boxed{40}$

$$W_r = U_{2r-1} + U_{2r}, \quad r \in \mathbb{Z}^+.$$

$$\therefore \sum_{r=1}^{n} W_r = \sum_{r=1}^{n} (U_{2r-1} + U_{2r})$$

$$= \sum_{r=1}^{2n} U_r \quad 5$$

$$= \frac{5}{144} - \frac{4n+5}{6(2n+1)(2n+2)(2n+3)(2n+4)}$$

$$\therefore \sum_{r=1}^{n} W_r = \frac{5}{144} - \frac{4n+5}{24(n+1)(n+2)(2n+1)(2n+3)} \quad 5$$

$\boxed{10}$

Note that,

$$\lim_{n\to\infty} \sum_{r=1}^{n} W_r = \lim_{n\to\infty}\left(\frac{5}{144} - \frac{4n+5}{24(n+1)(n+2)(2n+1)(2n+3)}\right) \quad 5$$

$$= \frac{5}{144} - \lim_{n\to\infty} \frac{4n+5}{24(n+1)(n+2)(2n+1)(2n+3)}$$

$$= \frac{5}{144} \quad 5$$

$\therefore \sum_{r=1}^{\infty} W_r$ is convergent and the sum is $\frac{5}{144}$. 5

$\boxed{15}$

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13. (*a*) Let $\mathbf{A} = \begin{pmatrix} a & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} 2 & 1 & 3 \\ 1 & -a & 4 \end{pmatrix}$ and $\mathbf{C} = \begin{pmatrix} b & -2 \\ -1 & b+1 \end{pmatrix}$ be matrices such that $\mathbf{AB}^{\mathrm{T}} = \mathbf{C}$, where $a, b \in \mathbb{R}$.

Show that $a = 2$ and $b = 1$.

Show also that, $\mathbf{C}^{-1}$ **does not** exist.

Let $\mathbf{P} = \frac{1}{2}(\mathbf{C} - 2\mathbf{I})$. Write down $\mathbf{P}^{-1}$ and find the matrix $\mathbf{Q}$ such that $2\mathbf{P}(\mathbf{Q} + 3\mathbf{I}) = \mathbf{P} - \mathbf{I}$, where $\mathbf{I}$ is the identity matrix of order 2.

(*b*) Let $z, z_1, z_2 \in \mathbb{C}$.

Show that (i) $\operatorname{Re} z \leq |z|$, and

(ii) $\left|\dfrac{z_1}{z_2}\right| = \dfrac{|z_1|}{|z_2|}$ for $z_2 \neq 0$.

Deduce that $\operatorname{Re}\left(\dfrac{z_1}{z_1 + z_2}\right) \leq \dfrac{|z_1|}{|z_1 + z_2|}$ for $z_1 + z_2 \neq 0$.

Verify that $\operatorname{Re}\left(\dfrac{z_1}{z_1 + z_2}\right) + \operatorname{Re}\left(\dfrac{z_2}{z_1 + z_2}\right) = 1$ for $z_1 + z_2 \neq 0$,

and show that $|z_1 + z_2| \leq |z_1| + |z_2|$ for $z_1, z_2 \in \mathbb{C}$.

(*c*) Let $\omega = \frac{1}{2}\left(1 - \sqrt{3}\,i\right)$.

Express $1 + \omega$ in the form $r(\cos\theta + i\sin\theta)$; where $r(>0)$ and $\theta\left(-\frac{\pi}{2} < \theta < \frac{\pi}{2}\right)$ are constants to be determined.

Using De Moivre's theorem, show that $(1+\omega)^{10} + (1+\bar{\omega})^{10} = 243$.

(*a*) $AB^T = \begin{pmatrix} a & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 1 & -a \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} 2a-3 & a-4 \\ -1 & a \end{pmatrix}$

5 10

$AB^T = C \quad \Leftrightarrow \quad \begin{pmatrix} 2a-3 & a-4 \\ -1 & a \end{pmatrix} = \begin{pmatrix} b & -2 \\ -1 & b+1 \end{pmatrix}$

$\Leftrightarrow\ 2a - 3 = b, \quad a - 4 = -2$ and $a = b + 1.$ 10

$\Leftrightarrow\ a = 2$ and $b = 1$, (from any two equations above) and these values satisfy the remaining equation. 5

$\boxed{30}$

$$C = \begin{pmatrix} 1 & -2 \\ -1 & 2 \end{pmatrix}$$

$$\begin{vmatrix} 1 & -2 \\ -1 & 2 \end{vmatrix} = 0 \quad 5$$

$\therefore\ C^{-1}$ does not exist. 5

10

Aliter

For the existence of C^{-1} :

there must exist $p, q, r, s \in \mathbb{R}$ such that

$$\begin{pmatrix} 1 & -2 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} p & q \\ r & s \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad 5$$

$\Rightarrow p - 2r = 1,\ -p + 2r = 0,\ q - 2s = 0$ and $-q + 2s = 1$

This is a contradiction

$\therefore\ C^{-1}$ does not exist. 5

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$$P = \frac{1}{2}(C - 2I) = \frac{1}{2}\left\{\begin{pmatrix} 1 & -2 \\ -1 & 2 \end{pmatrix} - \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}\right\} = \frac{1}{2}\begin{pmatrix} -1 & -2 \\ -1 & 0 \end{pmatrix} \quad 5$$

$$\Rightarrow P^{-1} = 2\left(\frac{1}{-2}\right)\begin{pmatrix} 0 & 2 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} 0 & -2 \\ -1 & 1 \end{pmatrix} \quad 10$$

$$2P(Q + 3I) = P - I$$

$$\Leftrightarrow \quad 2(Q + 3I) = I - P^{-1} \quad 5$$

$$\therefore\ 2(Q + 3I) = \begin{pmatrix} 1 & 2 \\ 1 & 0 \end{pmatrix} \quad 5$$

$$\Rightarrow Q = \frac{1}{2}\begin{pmatrix} 1 & 2 \\ 1 & 0 \end{pmatrix} - 3I$$

$$= \begin{pmatrix} -\frac{5}{2} & 1 \\ \frac{1}{2} & -3 \end{pmatrix} \quad 5$$

30

(*b*) $z,\ z_1,\ z_2 \in \mathbb{C}$.

(i) Let $z = x + iy,\ \ x, y \in \mathbb{R}$.

$\text{Re}\, z = x \leq \sqrt{x^2 + y^2} = |z|$ 5

(ii) Let $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$.

$$\Rightarrow \frac{z_1}{z_2} = \frac{r_1(\cos\theta_1 + i\sin\theta_1)\times(\cos\theta_2 - i\sin\theta_2)}{r_2(\cos\theta_2 + i\sin\theta_2)\times(\cos\theta_2 - i\sin\theta_2)} = \frac{r_1}{r_2}\ \frac{\left[\cos(\theta_1-\theta_2) + i\sin(\theta_1-\theta_2)\right]}{1}$$

5 5

$$\therefore \left|\frac{z_1}{z_2}\right| = \frac{r_1}{r_2} = \frac{|z_1|}{|z_2|}$$ 5

20

$$\text{Re}\left(\frac{z_1}{z_1+z_2}\right) \leq \left|\frac{z_1}{z_1+z_2}\right| = \frac{|z_1|}{|z_1+z_2|}$$; for $z_1 + z_2 \neq 0$.

5 by (i) 5 by (ii)

10

For $z_1 + z_2 \neq 0$, we have

$$\frac{z_1}{z_1+z_2} + \frac{z_2}{z_1+z_2} = 1$$ 5

$$\text{Re}\left(\frac{z_1}{z_1+z_2} + \frac{z_2}{z_1+z_2}\right) = 1$$

$$\text{Re}\left(\frac{z_1}{z_1+z_2}\right) + \text{Re}\left(\frac{z_2}{z_1+z_2}\right) = 1$$ 5

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$$\Rightarrow \quad 1 = \mathrm{Re}\left(\frac{z_1}{z_1+z_2}\right) + \mathrm{Re}\left(\frac{z_2}{z_1+z_2}\right) \le \left|\frac{z_1}{z_1+z_2}\right| + \left|\frac{z_2}{z_1+z_2}\right| \quad \text{by (i)} \quad 5$$

$$= \frac{|z_1|}{|z_1+z_2|} + \frac{|z_2|}{|z_1+z_2|} \quad \text{by (ii)}$$

$$= \frac{|z_1| + |z_2|}{|z_1+z_2|} \quad 5$$

$$\Rightarrow \quad |z_1 + z_2| \le |z_1| + |z_2| \qquad (\because |z_1+z_2| > 0)$$

Now if $z_1 + z_2 = 0$, then

$$|z_1 + z_2| = 0 \le |z_1| + |z_2|$$

Hence, the result is true for all $z_1, z_2 \in \mathbb{C}$.

$\boxed{10}$

(c) $\omega = \frac{1}{2}(1 - \sqrt{3}\, i)$

$$1 + \omega = \sqrt{3}\left[\frac{\sqrt{3}}{2} + i\left(\frac{-1}{2}\right)\right] = r(\cos\theta + i\sin\theta), \quad 5$$

where $r = \sqrt{3}$ and $\theta = -\frac{\pi}{6}$. $\quad 5$

$\boxed{10}$

$$(1+\omega)^{10} = (\sqrt{3})^{10}\left[\cos(10\theta) + i\sin(10\theta)\right] \text{ by De Moivre's theorem} \quad 5$$

$$1 + \overline{\omega} = \overline{1+\omega} = \sqrt{3}\,(\cos\theta - i\sin\theta) = \sqrt{3}\left[\cos(-\theta) + i\sin(-\theta)\right]$$

$$\Rightarrow (1+\overline{\omega})^{10} = \left(\sqrt{3}\right)^{10}\left[(\cos(-10\theta) + i\sin(-10\theta)\right] \quad 5$$

$$\therefore (1+\omega)^{10} + (1+\overline{\omega})^{10} = \left(\sqrt{3}\right)^{10} \times 2\cos(10\theta) \quad 5$$

$$= 3^5 \times 2 \times \frac{1}{2}$$

$$= 243. \quad 5$$

$\boxed{20}$

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14. (*a*) Let $f(x) = \dfrac{9(x^2 - 4x - 1)}{(x-3)^3}$ for $x \neq 3$.

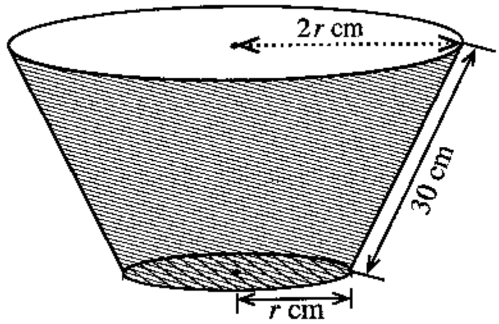
Show that $f'(x)$, the derivative of $f(x)$, is given by $f'(x) = -\dfrac{9(x+3)(x-5)}{(x-3)^4}$ for $x \neq 3$.

Sketch the graph of $y = f(x)$ indicating the asymptotes, y-intercept and the turning points.

It is **given that** $f''(x) = \dfrac{18(x^2 - 33)}{(x-3)^5}$ for $x \neq 3$.

Find the x-coordinates of the points of inflection of the graph of $y = f(x)$.

(*b*) The adjoining figure shows a basin in the form of a frustum of a right circular cone with a bottom. The slant length of the basin is 30 cm and the radius of the upper circular edge is twice the radius of the bottom. Let the radius of the bottom be r cm.



Show that the volume V cm^3 of the basin is given by $V = \frac{7}{3}\pi r^2\sqrt{900 - r^2}$ for $0 < r < 30$.

Find the value of r such that volume of the basin is maximum.

(*a*) For $x \neq 3$; $f(x) = \dfrac{9(x^2 - 4x - 1)}{(x-3)^3}$

Then

$$f'(x) = 9\left[\frac{1}{(x-3)^3}(2x-4) - \frac{3(x^2-4x-1)}{(x-3)^4}\right] \quad 20$$

$$= 9\left[\frac{2x^2 - 10x + 12 - 3(x^2 - 4x - 1)}{(x-3)^4}\right]$$

$$= \frac{-9(x+3)(x-5)}{(x-3)^4} \quad \text{for } x \neq 3 \quad 5$$

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Horizontal asymptotes : $\lim\limits_{x \to \pm\infty} f(x) = 0 \quad \therefore\ y = 0.$ 5

$\lim\limits_{x \to 3^-} f(x) = \infty$ and $\lim\limits_{x \to 3^+} f(x) = -\infty$.

Vertical asymptote : $x = 3$. 5

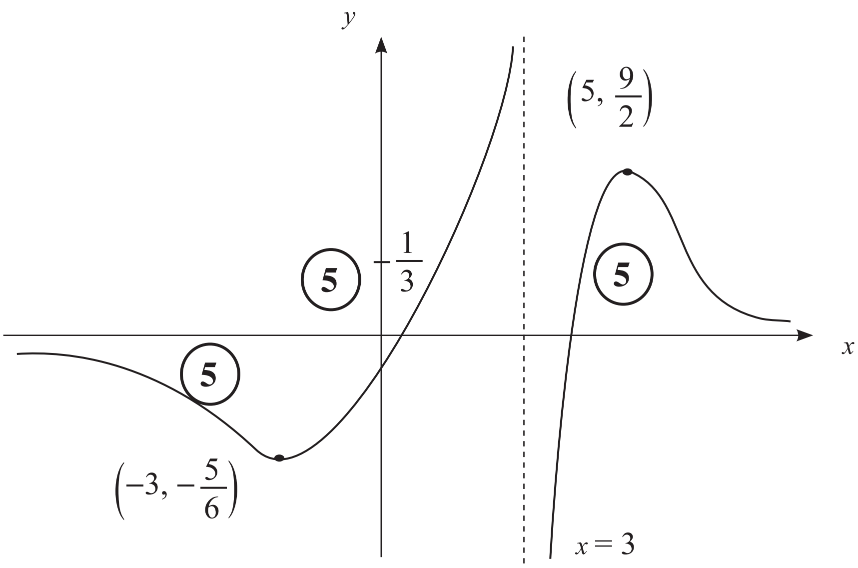
At the turning points $f'(x) = 0. \Leftrightarrow x = -3$ or $x = 5$. 5

	$-\infty < x < -3$	$-3 < x < 3$	$3 < x < 5$	$5 < x < \infty$
sign of $f'(x)$	$(-)$	$(+)$	$(+)$	$(-)$

5 5 5 5

$f(x)$ is ↘ ↗ ↗ ↘

There are two turning points: $\left(-3, -\frac{5}{6}\right)$ is a local minimum and $\left(5, \frac{9}{2}\right)$ is a local maximum. 5 5



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For $x \neq 3$;

$$f''(x) = \frac{18\,(x-\sqrt{33}\,)\,(x+\sqrt{33})}{(x-3)^5} \;.$$

$f''(x) = 0 \Leftrightarrow x = \pm\sqrt{33}\,.$ 5

	$-\infty < x < -\sqrt{33}$	$-\sqrt{33} < x < 3$	$3 < x < \sqrt{33}$	$\sqrt{33} < x < \infty$
sign of $f''(x)$	$(-)$	$(+)$	$(-)$	$(+)$
concavity	concave down	concave up	concave down	concave up

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$\therefore$ There are two inflection points:

$x = -\sqrt{33}$ and $x = \sqrt{33}$ are the x– coordinates of the points of inflection. 5

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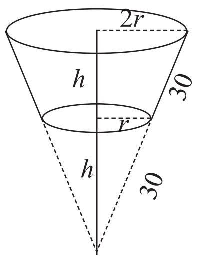
(***b***)



For $0 < r < 30$;

$h = \sqrt{900 - r^2}$ 5

The volume V is given by

$$V = \frac{1}{3}\pi (2r)^2 \times 2h - \frac{1}{3}\pi r^2 h \quad 5$$

$$= \frac{7}{3}\pi r^2 h$$

$$= \frac{7}{3}\pi r^2 \sqrt{900 - r^2}\,. \quad 5$$

$\boxed{15}$

For $0 < r < 30$,

$$\frac{dV}{dr} = \frac{7}{3}\pi \left[2r\sqrt{900 - r^2} + r^2 \frac{(-2r)}{2\sqrt{900 - r^2}}\right] \quad 5$$

$$= \frac{7}{3}\pi \left[\frac{2r(900 - r^2) - r^3}{\sqrt{900 - r^2}}\right]$$

$$= 7\pi r \frac{(600 - r^2)}{\sqrt{900 - r^2}}\,. \quad 5$$

$$\frac{dV}{dr} = 0 \Leftrightarrow r = 10\sqrt{6} \quad (\because r > 0) \quad 5$$

For $0 < r < 10\sqrt{6}$, $\frac{dV}{dr} > 0$ and for $r > 10\sqrt{6}$, $\frac{dV}{dr} < 0$

5 5

$\therefore V$ is maximum when $r = 10\sqrt{6}$. 5

$\boxed{30}$

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15. (*a*) Using the substitution $x = 2\sin^2\theta + 3$ for $0 \le \theta \le \frac{\pi}{4}$, evaluate $\int_3^4 \sqrt{\frac{x-3}{5-x}}\,dx$.

(*b*) Using partial fractions, find $\int \frac{1}{(x-1)(x-2)}\,dx$.

Let $f(t) = \int_3^t \frac{1}{(x-1)(x-2)}\,dx$ for $t > 2$.

Deduce that $f(t) = \ln(t-2) - \ln(t-1) + \ln 2$ for $t > 2$.

Using integration by parts, find $\int \ln(x-k)\,dx$, where k is a real constant.

Hence, find $\int f(t)\,dt$.

(*c*) Using the formula $\int_a^b f(x)\,dx = \int_a^b f(a+b-x)\,dx$, where a and b are constants,

show that $\int_{-\pi}^{\pi} \frac{\cos^2 x}{1+e^x}\,dx = \int_{-\pi}^{\pi} \frac{e^x \cos^2 x}{1+e^x}\,dx$.

Hence, find the value of $\int_{-\pi}^{\pi} \frac{\cos^2 x}{1+e^x}\,dx$.

(*a*) For $0 \le \theta \le \frac{\pi}{4}$:

$x = 2\sin^2\theta + 3 \Rightarrow dx = 4\sin\theta\cos\theta\,d\theta$ 5

$x = 3 \Leftrightarrow 2\sin^2\theta = 0 \Leftrightarrow \theta = 0$ 5

$x = 4 \Leftrightarrow 2\sin^2\theta = 1 \Leftrightarrow \sin\theta = \frac{1}{\sqrt{2}} \Leftrightarrow \theta = \frac{\pi}{4}$ 5

Then $\int_3^4 \sqrt{\frac{x-3}{5-x}}\,dx = \int_0^{\frac{\pi}{4}} \sqrt{\frac{2\sin^2\theta}{2-2\sin^2\theta}} \cdot 4\sin\theta\cos\theta\,d\theta$ 5

$= \int_0^{\frac{\pi}{4}} 4\sin^2\theta\,d\theta$ 5

$= 2\int_0^{\frac{\pi}{4}} (1-\cos 2\theta)\,d\theta$ 5

$= 2\left(\theta - \frac{1}{2}\sin 2\theta\right)\Big|_0^{\frac{\pi}{4}}$ 5

$= \frac{\pi}{2} - 1$ 5

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(*b*) $\dfrac{1}{(x-1)(x-2)} = \dfrac{A}{(x-1)} + \dfrac{B}{(x-2)}$

$\Leftrightarrow 1 = A(x-2) + B(x-1)$ for $x \neq 1, 2.$

Comparing coefficients of powers of x :

x^1 : $A + B = 0$ 5

x^0 : $-2A - B = 1$ 5

$A = -1$ and $B = 1$ 5

Then $\displaystyle\int \frac{1}{(x-1)(x-2)}\,\mathrm{d}x = \int \frac{-1}{(x-1)}\,\mathrm{d}x + \int \frac{1}{(x-2)}\,\mathrm{d}x$ 10

$= \ln|x-2| - \ln|x-1| + C$, where C is an arbitrary constant.

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$f(t) = \displaystyle\int_3^t \frac{1}{(x-1)(x-2)}\,\mathrm{d}x$

$= \left(\ln|x-2| - \ln|x-1|\right)\Big|_3^t$ 5

$= \ln(t-2) - \ln(t-1) + \ln 2$ for $t > 2.$ 5

10

$\displaystyle\int \ln(x-k)\,\mathrm{d}x = x\ln(x-k) - \int \frac{x}{(x-k)}\,\mathrm{d}x$ 5

$= x\ln(x-k) - \displaystyle\int 1\,\mathrm{d}x - \int \frac{k}{(x-k)}\,\mathrm{d}x$ 5

$= x\ln(x-k) - x - k\ln(x-k) + C$ 5

$= (x-k)\ln(x-k) - x + C$, where C is an arbitrary constant.

15

$\displaystyle\int f(t)\,\mathrm{d}t = \int \ln(t-2)\,\mathrm{d}t - \int \ln(t-1)\,\mathrm{d}t + \int \ln 2\,\mathrm{d}t$ 5

$= (t-2)\ln(t-2) - t - \Big[(t-1)\ln(t-1) - t\Big] + t\ln 2 + D$

$= (t-2)\ln(t-2) - (t-1)\ln(t-1) + t\ln 2 + D$, where D is an arbitrary constant.

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(*c*) Using the formula $\displaystyle\int_a^b f(x)\,\mathrm{d}x = \int_a^b (a+b-x)\,\mathrm{d}x,$

$$\int_{-\pi}^{\pi} \frac{\cos^2 x}{1+e^x}\,\mathrm{d}x = \int_{-\pi}^{\pi} \frac{\cos^2(-x)}{1+e^{-x}}\,\mathrm{d}x \quad 5$$

$$= \int_{-\pi}^{\pi} \frac{e^x\cos^2 x}{1+e^x}\,\mathrm{d}x \quad 5$$

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$$2\int_{-\pi}^{\pi} \frac{\cos^2 x}{1+e^x}\,\mathrm{d}x = \int_{-\pi}^{\pi} \frac{\cos^2 x}{1+e^{-x}}\,\mathrm{d}x + \int_{-\pi}^{\pi} \frac{e^x\cos^2 x}{1+e^x}\,\mathrm{d}x \quad 5$$

$$= \int_{-\pi}^{\pi} \frac{(1+e^x)\cos^2 x}{(1+e^x)}\,\mathrm{d}x$$

$$= \int_{-\pi}^{\pi} \cos^2 x\,\mathrm{d}x \quad 5$$

$$= \frac{1}{2}\int_{-\pi}^{\pi} (1+\cos 2x)\,\mathrm{d}x \quad 5$$

$$= \frac{1}{2}\left[x + \frac{1}{2}\sin 2x\right]_{-\pi}^{\pi} \quad 5$$

$$\therefore \int_{-\pi}^{\pi} \frac{\cos^2 x}{1+e^x} = \frac{\pi}{2} \quad 5$$

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16. Write down the coordinates of the point of intersection A of the straight lines $12x-5y-7=0$ and $y=1$.

Let l be the bisector of the acute angle formed by these lines. Find the equation of the straight line l.

Let P be a point on l. Show that the coordinates of P can be written as $(3\lambda+1,\ 2\lambda+1)$, where $\lambda\in\mathbb{R}$.

Let $B\equiv(6,0)$. Show that the equation of the circle with the points B and P as ends of a diameter can be written as $S+\lambda U=0$, where $S\equiv x^2+y^2-7x-y+6$ and $U\equiv-3x-2y+18$.

Deduce that $S=0$ is the equation of the circle with AB as a diameter.

Show that $U=0$ is the equation of the straight line through B, perpendicular to l.

Find the coordinates of the fixed point which is distinct from B, and lying on the circles with the equation $S+\lambda U=0$ for all $\lambda\in\mathbb{R}$.

Find the value of λ such that the circle given by $S=0$ is orthogonal to the circle given by $S+\lambda U=0$.

$12x-5y-7=0$ and $y=1 \Rightarrow x=1,\quad y=1$ 10

$\therefore\ A \equiv (1,1)$

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Equations of the bisectors are given by

$$\frac{12x-5y-7}{13} = \pm\frac{(y-1)}{1}$$ 10

$\Rightarrow\ 12x-5y-7 = 13(y-1)$ or $12x-5y-7 = -13(y-1)$

$\Rightarrow\ 2x-3y+1 = 0$ or $3x+2y-5 = 0$ 5 + 5

The angle θ between $y=1$ and $2x-3y+1=0$, is given by

$$\tan\theta = \left|\frac{\frac{2}{3}-0}{1+\frac{2}{3}(0)}\right| = \frac{2}{3} < 1$$ 5

$\therefore\ l:\ 2x-3y+1=0.$ 5

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Note that for a point (x, y) on l ;

$$\frac{(x-1)}{3} = \frac{(y-1)}{2} = \lambda \text{ (say)} \quad 5$$

$\Rightarrow x = 3\lambda + 1, \quad y = 2\lambda + 1.$ 5

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$\therefore \; P \equiv (3\lambda + 1, \; 2\lambda + 1), \quad \lambda \in \mathbb{R}.$

Note that $B \equiv (6, 0)$ and $P \equiv (3\lambda + 1, \; 2\lambda + 1)$

$\therefore$ Equation of the circle with BP as a diameter is given by

$(x-6)\;(x - (3\lambda + 1)) + (y - 0)\;(y - (2\lambda + 1)) = 0$ 10

i.e. $(x^2 + y^2 - 7x - y + 6) + \lambda\,(-3x - 2y + 18) = 0$ 5

This is of the form $S + \lambda U = 0$, where $S \equiv x^2 + y^2 - 7x - y + 6$ and $U \equiv -3x - 2y + 18$.

5 5 25

$S = 0$ corresponds to $\lambda = 0$. $\Rightarrow P = (1, 1) \equiv A$. 5

$\therefore$ $S = 0$ is the equation of the circle with AB as a diameter. 5 10

Since the slope of l is $\frac{2}{3}$, the equation of the line perpendicular to l passing through B is $3x + 2y + \mu = 0$, μ to be determined. 10

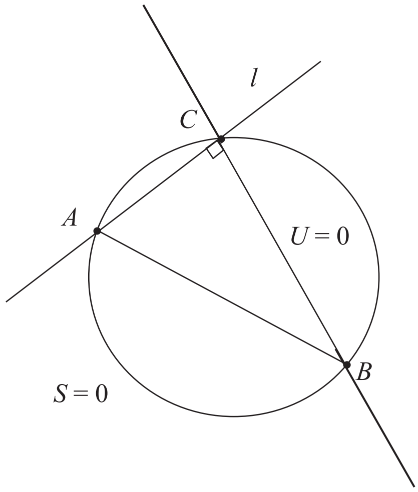
Since B lies on $3x + 2y + \mu = 0$, we have $18 + \mu = 0 \Rightarrow \mu = -18$. 5

$\therefore$ Required equation is $3x + 2y - 18 = 0$ 5

i.e. $U = -3x - 2y + 18 = 0$. 20

$\lambda \in \mathbb{R}$, $S + \lambda U = 0$ passes through the intersection point of $S = 0$ and $U = 0$ 10

One of these points is B and the other point C is the intersection point of l and $U = 0$. 10



$\therefore$ The coordinates of C is given by

$$u \equiv -3x - 2y + 18 = 0$$

and $l \equiv 2x - 3y + 1 = 0$

$\Rightarrow$ $x = 4$ and $y = 3$

$\therefore$ $C \equiv (4, 3)$. 5

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The circles ;

$S = 0$ and $S + \lambda U = 0$ are orthogonal

$$\Leftrightarrow \quad 2\left(-\frac{1}{2}(3\lambda + 7)\right)\left(-\frac{7}{2}\right) + 2\left(-\frac{1}{2}(2\lambda + 1)\right)\left(-\frac{1}{2}\right) = 6 + 18\lambda + 6$$

5 5 5

$\Leftrightarrow$ $13\lambda = 26$ 5

$\Leftrightarrow$ $\lambda = 2.$

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17. (*a*) Write down $\sin(A+B)$ in terms of $\sin A$, $\cos A$, $\sin B$ and $\cos B$, and obtain a similar expression for $\sin(A-B)$.

Deduce that

$$2\sin A\cos B = \sin(A+B)+\sin(A-B) \text{ and}$$
$$2\cos A\sin B = \sin(A+B)-\sin(A-B).$$

Hence, solve $2\sin 3\theta\cos 2\theta = \sin 7\theta$ for $0 < \theta < \frac{\pi}{2}$.

(*b*) In a triangle ABC, the point D lies on AC such that $BD = DC$ and $AD = BC$. Let $B\hat{A}C = \alpha$ and $A\hat{C}B = \beta$. Using the Sine Rule for suitable triangles, show that $2\sin\alpha\cos\beta = \sin(\alpha+2\beta)$.

If $\alpha : \beta = 3 : 2$, using the last result in (*a*) above, show that $\alpha = \frac{\pi}{6}$.

(*c*) Solve $2\tan^{-1}x + \tan^{-1}(x+1) = \frac{\pi}{2}$. **Hence**, show that $\cos\left(\frac{\pi}{4} - \frac{1}{2}\tan^{-1}\left(\frac{4}{3}\right)\right) = \frac{3}{\sqrt{10}}$.

(***a***) $\sin(A+B) = \sin A\cos B + \cos A\sin B$ ——— (1) **5**

Now $\sin(A-B) = \sin(A+(-B))$ **5**

$= \sin A\cos(-B) + \cos A\sin(-B)$

$\therefore \sin(A-B) = \sin A\cos B - \cos A\sin B$ ——— (2) **5**

$\boxed{15}$

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(1) + (2) $\Rightarrow \sin(A+B) + \sin(A-B) = 2\sin A\cos B$, **5**

(1) − (2) $\Rightarrow \sin(A+B) - \sin(A-B) = 2\cos A\sin B$. **5**

$\boxed{10}$

$\boxed{0 < \theta < \frac{\pi}{2}.}$

$2\sin 3\theta\cos 2\theta = \sin 7\theta$,

$\Leftrightarrow \sin 5\theta + \sin\theta = \sin 7\theta$ **5**

$\Leftrightarrow \sin 7\theta - \sin 5\theta - \sin\theta = 0$

$\Leftrightarrow \sin(6\theta+\theta) - \sin(6\theta-\theta) - \sin\theta = 0$

$\Leftrightarrow 2\cos 6\theta\sin\theta - \sin\theta = 0$ **5**

$\Leftrightarrow \sin\theta(2\cos 6\theta - 1) = 0$

$\Leftrightarrow \cos 6\theta = \frac{1}{2}$ since $0 < \theta < \frac{\pi}{2}$, $\sin\theta > 0$

5

$$\Rightarrow \ 6\theta = 2n\pi \pm \frac{\pi}{3} \ ; \ n \in \mathbb{Z}. \quad 5 + 5$$

$$\Rightarrow \ \theta = \frac{n\pi}{3} \pm \frac{\pi}{18} \ ; \ n \in \mathbb{Z}.$$

$$\Rightarrow \ \theta = \frac{\pi}{18}, \frac{5\pi}{18}, \frac{7\pi}{18}, \left(\because \ 0 < \theta < \frac{\pi}{2}\right) \quad 5$$

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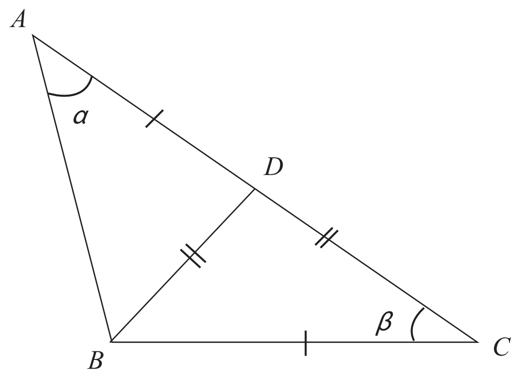
(b)



Note that

$C\hat{B}D = \beta, \ A\hat{D}B = 2\beta,$

and $A\hat{B}D = \pi - (\alpha + 2\beta)$

Using the sine Rule :

for the triangle ABD, we have

$$\frac{BD}{\sin B\hat{A}D} = \frac{AD}{\sin A\hat{B}D} \quad 10$$

$$\Rightarrow \ \frac{BD}{\sin \alpha} = \frac{AD}{\sin (\pi - (\alpha + 2\beta))}$$

$$\Rightarrow \ \frac{BD}{\sin \alpha} = \frac{AD}{\sin (\alpha + 2\beta)} \quad 5 \ \text{———} \ (1)$$

for the triangle BDC, we have

$$\frac{CD}{\sin D\hat{B}C} = \frac{BC}{\sin B\hat{D}C} \quad 10$$

$$\Rightarrow \ \frac{CD}{\sin \beta} = \frac{BC}{\sin 2\beta} \ \text{———} \ (2) \quad 5$$

$\because \ BD = DC$ and $AD = BC$, from (1) and (2), we get

$$\frac{\sin \alpha}{\sin \beta} = \frac{\sin (\alpha + 2\beta)}{\sin 2\beta} \quad 5$$

$\Rightarrow \quad 2\sin\alpha\cos\beta = \sin(\alpha + 2\beta).$ 5

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If $\alpha : \beta = 3 : 2$, then we have

$2\sin\alpha\,\cos\dfrac{2\alpha}{3} = \sin\dfrac{7\alpha}{3}$ 5

$\Rightarrow \quad 2\sin 3\left(\dfrac{\alpha}{3}\right)\cos 2\left(\dfrac{\alpha}{3}\right) = \sin 7\left(\dfrac{\alpha}{3}\right)$ 5

$\Rightarrow \quad \dfrac{\alpha}{3} = \dfrac{\pi}{18}, \dfrac{5\pi}{18}, \dfrac{7\pi}{18}.$

$\Rightarrow \quad \alpha = \dfrac{\pi}{6}, \dfrac{15\pi}{18}, \dfrac{21\pi}{18}$ 5

$\because\ BC = AD < AC$, α must be an acute angle.

$\therefore\ \alpha = \dfrac{\pi}{6}.$ 5

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(c) $2\tan^{-1}x + \tan^{-1}(x+1) = \dfrac{\pi}{2}$

Let $\alpha = \tan^{-1}(x)$ and $\beta = \tan^{-1}(x+1)$. Note that $x \neq \pm 1$.

Then $2\alpha + \beta = \dfrac{\pi}{2}.$ 5

$\Leftrightarrow \quad 2\alpha = \dfrac{\pi}{2} - \beta$

$\Leftrightarrow \quad \tan 2\alpha = \tan\left(\dfrac{\pi}{2} - \beta\right)$ 5

$\Leftrightarrow \quad \dfrac{2\tan\alpha}{1-\tan^2\alpha} = \cot\beta$ 5 + 5

$\Leftrightarrow \quad \dfrac{2x}{1-x^2} = \dfrac{1}{x+1}$ 5

$\Leftrightarrow \quad 2x = 1 - x \quad (\because\ x \neq \pm 1)$

$\Leftrightarrow \quad x = \dfrac{1}{3}.$ 5

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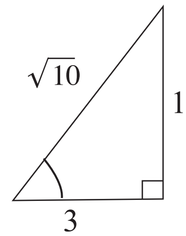
Note that

$2\tan^{-1}\left(\frac{1}{3}\right) + \tan^{-1}\left(\frac{4}{3}\right) = \frac{\pi}{2}$.

$\Rightarrow \frac{\pi}{4} - \frac{1}{2}\tan^{-1}\left(\frac{4}{3}\right) = \tan^{-1}\left(\frac{1}{3}\right)$



$\Rightarrow \cos\left(\left(\frac{\pi}{4}\right) - \frac{1}{2}\tan^{-1}\left(\frac{4}{3}\right)\right) = \cos\left(\tan^{-1}\left(\frac{1}{3}\right)\right)$ 5

$\therefore \cos\left(\frac{\pi}{4} - \frac{1}{2}\tan^{-1}\left(\frac{4}{3}\right)\right) = \frac{3}{\sqrt{10}}$ 5

$\boxed{10}$