(07) Mathematics

Structure of the Question Paper

Paper I - **Time : 03 hours** (In addition, 10 minutes for reading.)
This paper consists of **two** parts.
Part A : Ten questions. **All** questions should be answered. 25 marks for each question - altogether 250 marks.
Part B : Seven questions. **Five** questions should be answered. Each question carries 150 marks - altogether 750 marks.
Total marks for paper I = 1000

Paper II - **Time : 03 hours** (In addition, 10 minutes for reading.)
This paper consists of **two** parts.
Part A : Ten questions. **All** questions should be answered. 25 marks for each question - altogether 250 marks.
Part B : Seven questions. **Five** questions should be answered. Each question carries 150 marks - altogether 750 marks.
Total marks for paper II = 1000

Calculation of the final mark :

Paper I	=	1000 ÷ 20 = 50
Paper II	=	1000 ÷ 20 = 50
Final mark	=	100

(07) Mathematics

Paper I

Part A

1. Let $A = \{x \in \mathbb{R} : \; | x + 3 | < 2 \}$ and $B = \{x \in \mathbb{R} : | x | \geq 4\}$ be subsets of the universal set $\mathbb{R}$. Find $A \cap B$ and $A' \cap B$.

2. Let A and B be subsets of a universal set S. The set $A \setminus B$ is defined, in the usual notation, by $A \setminus B = A \cap B'$. Show that $A \setminus (B \cup C) = (A \setminus B) \cap (A \setminus C)$ and $A \setminus (B \cap C) = (A \setminus B) \cup (A \setminus C)$.

3. Let p and q be propositions. Show that the compound propositions $\sim(p \vee (\sim p \wedge q))$ and $\sim p \wedge \sim q$ are logically equivalent.

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4. Using the method of **proof by contradiction**, prove that if $3n^2 + 2$ is odd, then n is odd.

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5. Solve the simultaneous equations $y - \frac{1}{3}\log_2 x = 0$ and $8^{2y-1} - 2(x-4) = 0$ for x and y.

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6. Find all real values of x satisfying the inequality $x - \frac{4}{x} \leq 3$.

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7. Let $f(x) = \sqrt{x+3} - 5$ be a function defined on $[-3,\infty)$. Find the range of the function f and show that f is one-to-one. Find $f^{-1}(x)$.

8. The line l has gradient -3 and passes through the point $A(2,1)$. A point B is on the line l such that the distance AB is $3\sqrt{10}$. Find the possible coordinates for the point B.

9. Find the points at which the tangents to the parametric curve given by $x = 2t^3$, $y = 2 - 4t + t^2$ has a slope of -1.

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10. Find the area of the region bounded by the curves $y = x^2$ and $x + y = 2$.

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Part B

11. (*a*) Fifty students sat for an examination in the subjects, Mathematics, Physics and Chemistry. Out of these 50 students, 37 passed Mathematics, 24 passed Physics and 43 passed Chemistry. Further, it is given that at most 19 students passed Mathematics and Physics, at most 29 passed Mathematics and Chemistry and at most 20 passed Physics and Chemistry. Find the largest possible number of students that could have passed all three subjects.

(*b*) Determine whether the compound proposition $[\sim p \wedge (p \vee q)] \rightarrow q$ is a tautology or a contradiction.

12. (*a*) Using the **Principle of Mathematical Induction**, prove that

$$\sum_{r=1}^{n}(3r^2+5r+1)=n(n+2)^2 \text{ for all } n\in\mathbb{Z}^+.$$

(*b*) Let $U_r=\dfrac{2}{(2r-1)(2r+1)}$ for $r\in\mathbb{Z}^+$.

Verify that $U_r=\dfrac{1}{(2r-1)}-\dfrac{1}{(2r+1)}$ for $n\in\mathbb{Z}^+$, and show that $\sum\limits_{r=1}^{n}U_r=\dfrac{2n}{2n+1}$ for $n\in\mathbb{Z}^+$.

Also, find $\sum\limits_{r=10}^{20}(2U_r+3r)$.

13. (*a*) The roots of the quadratic equation $x^2+(4+k)x-(25+k)=0$ are α and $-\alpha^2$, where k is a real constant.

Show that α is a root of the equation $x^3-x^2+x-21=0$.

Show that $(x-3)$ is a factor of x^3-x^2+x-21 and show that the equation $x^3-x^2+x-21=0$ has only one real root.
Hence, find the value of k.

(*b*) Let $f(x)=-2x^2+12x-16$.

Write the function $f(x)$ in the form $a(x-h)^2+k$, where a, h and k are constants to be determined.

Find the coordinates of the vertex, equation of the axis of symmetry, and the maximum value of f. Sketch the graph of the function $y=f(x)$.

The function g is defined by $g(x)=-2-f(x+1)$. Determine the axis of symmetry, and the minimum value of the function g.

14. (*a*) Write down, in the usual notation, the binomial expansion of $(a+b)^n$, where a and b real numbers and n is a positive integer.

(i) If the sum of the coefficients of the first, second and the third terms of the binomial expansion of $\left(x^2+\frac{1}{x}\right)^n$ is 46, find n.

(ii) Find the value of k if the coefficient of x^4 in the expansion of $\left(kx+\frac{1}{x}\right)^{10}$ is equal to $\frac{15}{16}$. For this value of k, find the term of the expansion that is independent of x.

(*b*) A person has the following 3 investment options:

Option 1: Invest under 14% simple interest per annum
Option 2: Invest under 12% compound interest per annum
Option 3: Invest under quarterly compounded 8% interest per annum

(i) Select the best investment option based on the interest accumulated at the end of 5 years.

(ii) The person also has the 4th option of investment where the interest is calculated quarterly at an annual rate of r%. If the interest under option 4 is larger than that is under option 2 for a period of 10 years, what is the minimum value of r?

15. Let $y = m_1 x + c_1$, $y = m_2 x + c_2$ and $x = 0$ be the equations of the sides AB, BC and AC of the triangle ABC respectively. Show that the area of triangle ABC is given by $\dfrac{(c_1 - c_2)^2}{2|m_1 - m_2|}$.

Let $3x - y + 5 = 0$, $2x + 3y - 1 = 0$ and $x + 2y - 3 = 0$ be the equations of the sides BC, CA and AB respectively of the triangle ABC.

A straight line passing through the point A with gradient $-\dfrac{1}{3}$ intersects at the point D with a straight line passing through the point B and parallel to CA. If O is the origin, show that the equation of OD is given by $y + x = 0$.

The straight line passing through the point D and perpendicular to the side AB meet the y -axis at the point E. Find the area of the triangle ODE.

16. (*a*) Find $\displaystyle\lim_{x \to 2} \frac{\sqrt{x} - \sqrt{2}}{x^2 - 4}$.

(*b*) Differentiate each of the following with respect to x.

(i) $\left(\dfrac{x}{1 - x}\right)^6$

(ii) $\dfrac{e^{2x} + e^{-2x}}{e^{2x} - e^{-2x}}$

(iii) $x^2 \ln (x^4 + 1)$

(*c*) An open tank of volume 4000 m^3 having a square base and vertical walls is to be constructed from thin sheet material. Find the dimensions of the tank such that the material used is a minimum.

17. (*a*) Using **integration by parts**, evaluate $\displaystyle\int_0^1 x^2 e^{2x}\, dx$

(*b*) Using **partial fractions**, find $\displaystyle\int \frac{2x + 3}{(x + 1)(x + 2)^2}\, dx$.

(*c*) The following table gives the values of the function $f(x) = \sqrt{2x + 1}$, correct to three decimal places for values of x between 0 and 1 at intervals of length 0.25.

x	0	0.25	0.50	0.75	1.00
$f(x)$	1	1.225	1.414	1.581	1.732

Using **Simpson's rule**, find an approximate value for $I = \int_0^1 \sqrt{2x + 1}\, dx$ correct to two decimal places.

Using the substitution $u = 2x + 1$, find I and compare the value of I with the approximate value obtained above.

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(07) Mathematics

Paper II

Part A

1. Find values of x satisfying $\begin{vmatrix} 1 & 1 & x \\ 4 & 4 & x+1 \\ 3 & x+1 & x+2 \end{vmatrix} = 0.$

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2. Let $A = \begin{pmatrix} 2 & 0 \\ 3 & -1 \\ 1 & 4 \end{pmatrix}$, $B = \begin{pmatrix} -1 & 1 \\ 0 & -3 \\ -2 & 1 \end{pmatrix}$ and $C = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$.

Find $A - 2B$, AC and BC. Verify that $(A - 2B)\,C = AC - 2BC$.

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3. Certain IQ test scores follow a normal distribution with mean of 100 and standard deviation of 16. Compute the cut off value that bounds the highest 5% of all IQ test scores.

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4. The mean, median and standard deviation of a particular distribution are 61, 52 and 10 respectively. Calculate the coefficient of skewness and comment on the shape of the distribution. Is mean a reasonable measure of central tendency for this distribution? Justify your answer.

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5. In a production process, biscuits are packed in two sizes, viz. 100 g and 200 g. Following summary measures were calculated based on testing done on samples of packets.

Size	**Sample size**	**Sample mean**	**Standard deviation**
100 g	20	102 g	2.5 g
200 g	20	203 g	3.1 g

Calculating coefficient of variation, determine the size of packet which is more consistent in weight.

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6. A continuous random variable X is uniformly distributed over the interval $[a, 6a]$, where a is a positive constant. Find the distribution function of X.

Another continuous random variable Y is uniformly distributed over the interval $[-2, 8]$. If $P(X < 3) = P(Y < 4)$, find the value of a.

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7. A certain type of knee surgery has a 75% chance of success. The surgery was performed on four patients. Find the probability of the surgery being successful on exactly two patients.

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8. The random variable X has the following probability distribution:

x	1	2	3	4	5
$P(X=x)$	p	0.2	q	0.3	0.1

If $E(X) = 3.1$, find p and q. Find Var(X).

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9. Let A and B be two events of a sample space S. If $P(A \cap B) = \frac{1}{5}$ and $P(A) = P(A|B') = \frac{7}{15}$, then find $P(B|A)$ and $P(B)$.

Determine whether the two events A and B are independent.

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10. The random variable X has probability density function $f(x)$ given by $f(x) = \begin{cases} x - k, & \text{if } 0 \le x \le 2, \\ 0, & \text{otherwise} \end{cases}$,
where k is a constant. Show that $k = \frac{1}{2}$ and find the mean of X.

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Part B

11. A company produces 2 types of products A and B, where each of the product should go through 2 different processes. The time required at each process to produce one unit of product depends on the type of product. The hours needed at each process to produce one unit of product A and B, and the number of hours of work that can be handled by each process per week is given in the table below.

		Time required (hours) per unit		**Number of hours of work that can be handled by the process per week**
		Process 1	**Process** 2	
Product	A	2	4	40
	B	4	4	32

Suppose that the company needs to produce at least 2 units from each of the products A and B.

The profit per unit of products A and B are 10 rupees and 5 rupees respectively. Assume that all units produced can be sold. It is required to determine the number of units to be produced per week from each product to maximize the total profit.

(*a*) Formulate this as a linear programming problem.
(*b*) Sketch the feasible region and **hence** solve the problem graphically.

12. (*a*) If $A = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}$ and $X = \begin{pmatrix} x & 2 \\ 3 & -y \end{pmatrix}$, find the values of x and y such that $AX = XB$.

(*b*) Let $A = \begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{pmatrix}$. Show that $A^2 - 4A = 5I$, where I is the identity matrix of order 3.

Hence or otherwise, find the square matrix B of order 3 such that $BA = I$.

Consider the following system of linear equations:

$$\begin{aligned} x + 2y + 2z &= -1, \\ 2x + y + 2z &= 2, \\ 2x + 2y + z &= -1. \end{aligned}$$

Taking $C = \begin{pmatrix} -1 \\ 2 \\ -1 \end{pmatrix}$ and $X = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$, show that the matrix equation $AX = C$ represents the above system of linear equations.

Hence, solve the above system of linear equations.

13. (*a*) Three cards are numbered as 1, 3, and 4. A game consists of picking one card at random and rolling a six sided fair die with faces numbered 1, 2, 3, 4, 5 and 6. Let the number on the card picked be x and let the number on the die facing up be y. The events A and B are defined as follows:

A : $x \geq y$,
B : $x + y$ is an even number.

(i) Find the $P(A)$, $P(B)$ and $P(A|B)$.

(ii) Determine whether the events A and B are mutually exclusive.

(*b*) (i) Find the number of different permutations that can be formed from the eleven letters of the word "COEFFICIENT".

(ii) Find the number of different combinations of four letters that can be formed from eleven letters of the word "COEFFICIENT".

14. (*a*) An ice- cream seller has to decide whether to order more stock for the holiday weekend. From past experience he knows that there is an 85% chance of selling all his stock, if the weather is sunny; if it is cloudy his chance is 65%; and if it rains, his chance is only 10%. According to weather forecast, the probability of sunny is 40%, the probability of cloudy is 35% and the probability of rainy is 25%.

(i) What is the probability that the seller will sell all his stock?

(ii) What is the probability that the weather was sunny, given that he sold all his stock of ice- cream?

(*b*) The Body Mass Index (BMI) is used to classify people as under-weight, normal-weight and over-weight. The classification is shown below.

Under-weight : if BMI ≤ 18.5
Normal-weight : if $18.5 <$ BMI < 25.0
Over-weight : if BMI ≥ 25.0

In a particular population, Body Mass Index (BMI) is normally distributed with mean 20 and standard deviation 4.

(i) Calculate the percentage of people belonging to each of the above weight categories.

(ii) If 200 people were randomly selected from the above described population, how many under-weight people can be expected among the selected people.

15. Assume that an insurance policy holder is two times more likely to file 2 claims as to file 3 claims per month. Suppose that the number of claims X of that policy holder in a month follows a Poisson distribution with probability mass function given by $P(X=x)=\dfrac{e^{-\lambda}\lambda^{x}}{x!}$, for $x=0,1,2,3,\ldots$.

(*a*) Find λ .

(*b*) Find the probability that the insurance policy holder files at least one claim per month (you may take $e^{-5} \cong 0.6065$).

(*c*) If the insurance policy holder continues to file claims in every month in a similar manner, find the expected number of claims that will be filed a year.

16. Monthly income figures of 50 families are summarized in the table below.

Income (Rupees)	**No. of families**
10 000 - 14 999	2
15 000 - 19 999	8
20 000 - 24 999	15
25 000 - 29 999	9
30 000 - 34 999	6
35 000 - 39 999	5
40 000 - 44 999	3
45 000 - 49 999	2

(i) Using a suitable coding method calculate mean, median and mode of the monthly income.

(ii) Estimate the inter-quartile range of the monthly income.

(iii) Families with monthly income less than Rs. 20 000 are considered as low income families. Calculate the percentage of low income families.

(iv) A subsidiary was given to all the low income families to bring up their monthly income upto Rs. 20 000. What is the inter-quartile range of the monthly income after giving this subsidiary?

17. The relationships between the activities of a project and the duration of each activities are given below.

Activity	**Immediate predecessor/s**	**Duration (in weeks)**
A	–	2
B	A	3
C	A	5
D	B	8
E	B, C	4
F	E	6
G	D, F	7
H	G	9

(i) Construct the project network.

(ii) Write down the critical activities of the project.

(iii) Prepare a time schedule for each activity including earliest start time, earliest finish time, latest start time, latest finish time and float.

(iv) What are the activities that cannot be delayed without extending the total duration of the project?

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